A pathological o-minimal quotient

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Abstract

We give an example of a definable quotient in an o-minimal structure which cannot be eliminated over any set of parameters, giving a negative answer to a question of Eleftheriou, Peterzil, and Ramakrishnan. Equivalently, there is an o-minimal structure M whose elementary diagram does not eliminate imaginaries. We also give a positive answer to a related question, showing that any imaginary in an o-minimal structure is interdefinable over an independent set of parameters with a tuple of real elements. This can be interpreted as saying that interpretable sets look "locally" like definable sets, in a sense which can be made precise.

1 Elimination of imaginaries and o-minimality

In o-minimal expansions of real closed fields, as well as many other o-minimal theories, elimination of imaginaries holds as a corollary of definable choice. As noted in [1], some o-minimal theories fail to eliminate imaginaries. For example, elimination of imaginaries fails in the theory of $\mathbb{Q}$ with the ordering and with a 4-ary predicate for the relation $x - y = z - w$. In [2], Eleftheriou, Peterzil, and Ramakrishnan observe that in this example, elimination of imaginaries holds after naming two parameters. This leads them to pose the following question:

Question 1.1. *Given an o-minimal structure M and a definable equivalence relation E on a definable set X, both definable over a parameter set A, is there a definable map which eliminates X/E, possibly over $B \supseteq A$?*

They answer this question in the affirmative when X/E has a definable group structure, as well as when $\dim(X/E) = 1$. However, we will answer Question 1.1 negatively by giving a counterexample in §2. That is, we will give an o-minimal structure M and a set X/E interpretable in M, which cannot be put in definable bijection with a definable subset of M^k.

Question 1.1 can be reformulated in several ways, by the following observation.

Lemma 1.2. *Let M be a structure, and let $\mathbb{M} \succeq M$ be any elementary extension, such as a monster model. The following are equivalent:*

(a) *Every M-definable quotient can be eliminated over M.*

(b) *Every M-definable quotient can be eliminated over $\mathbb{M}$.*

(c) *Every $\mathbb{M}$-definable quotient can be eliminated over $\mathbb{M}$.*

(d) *The elementary diagram of M eliminates imaginaries.*

Proof. The implications $(a) \Rightarrow (d) \Rightarrow (c) \Rightarrow (b)$ are more or less clear. For $(b) \Rightarrow (a)$, suppose (b) holds and X/E is an M-definable quotient. By (b), X/E can be eliminated by an $\mathbb{M}$-definable function f. Since M is an elementary substructure of $\mathbb{M}$, the parameters used to define f can be moved into M, so (a) holds. □

Question 1.1 asks whether the equivalent conditions of Remark 1.2 hold in every o-minimal structure M. We will give an example in which they fail.

In a talk at the 2012 Banff meeting on Neo-Stability, Peterzil asked the following variant of Question 1.1:

Question 1.3. *Given an o-minimal structure M and an imaginary $e \in M^{eq}$, is there a set $A \subset M$ and a real tuple $c \in M^k$ such that $A \mathop{\smile\!\!\!\!\!\!\mid}^{þ} e$ and $\mathrm{dcl}^{eq}(Ae) = \mathrm{dcl}^{eq}(Ac)$?*

Here $\mathop{\smile\!\!\!\!\!\!\mid}^{þ}$ denotes thorn-forking, or equivalently, independence with respect to o-minimal dimension.

In contrast to the negative answer to Question 1.1, we anser Question 1.3 positively in §3. In some sense, this suggests that interpretable sets, while not being "globally" definable, look "locally" like definable sets. We state a result in this direction, Theorem 3.2, without proof.

2 The counterexample

Let $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ be the real projective line. The group $PSL_2(\mathbb{R})$ acts on $\mathbb{RP}^1$ by fractional linear transformations, $x \mapsto \frac{ax+b}{cx+d}$, and the stabilizer of ∞ is exactly the group of affine transformations $x \mapsto ax + b$.

For $x, y_1, \ldots, y_4 \in \mathbb{RP}^1$, let $P_0(x, y_1, \ldots, y_4)$ indicate that $x \notin \{y_1, \ldots, y_4\}$ and that

$$f(y_1) - f(y_2) = f(y_3) - f(y_4)$$

for any/every fractional linear transformation f sending x to ∞. The choice of f does not matter, because if f and f' both send x to ∞, then $f' = h \circ f$ for some affine transformation h. But in general,

$$h(z_1) - h(z_2) = h(z_3) - h(z_4) \iff z_1 - z_2 = z_3 - z_4$$

for h affine.

Remark 2.1. *If g is some fractional linear transformation, then g induces an automorphism on the structure $(\mathbb{RP}^1, P_0)$. In particular, if $a > 0$ and $b \in \mathbb{R}$, then the map $x \mapsto ax + b$ (fixing ∞) is an automorphism.*

Remark 2.2. *Write* $\cot(x)$ *for* $1/\tan(x)$*. If* $\alpha \in \mathbb{R}$*, then* $-\cot(x)$ *and* $\cot(x-\alpha)$ *are related by a fractional linear transformation not depending on* x*, sending* $-\cot(\alpha)$ *to* $\cot(0) = \infty$*. Consequently, if* $\alpha, x_1, \ldots, x_4 \in \mathbb{R}$*, then*

$$\begin{aligned} &P_0(-\cot(\alpha), -\cot(x_1), \ldots, -\cot(x_4)) \\ &\iff \cot(x_1 - \alpha) - \cot(x_2 - \alpha) = \cot(x_3 - \alpha) - \cot(x_4 - \alpha). \end{aligned}$$

Let M be the structure $(\mathbb{Z} \times \mathbb{RP}^1, <, \sigma, P)$, where

- $<$ is the lexicographic order on $\mathbb{Z} \times \mathbb{RP}^1$, where we order $\mathbb{RP}^1$ by identifying it with $[-\infty, +\infty)$.
- σ is the map $(n, x) \mapsto (n+1, x)$.
- $P(x, y_1, \ldots, y_4)$ holds if and only if

$$P_0(\pi_2(x), \pi_2(y_1), \ldots, \pi_2(y_4)) \wedge \bigwedge_{i=1}^{4} x < y_i < \sigma(x)$$

  where $\pi_2 : \mathbb{Z} \times \mathbb{RP}^1 \to \mathbb{RP}^1$ is the second coordinate projection.

Remark 2.3. *If* $a > 0$ *and* $b \in \mathbb{R}$*, then the map* $(n, x) \mapsto (n, ax + b)$*, fixing* (n, ∞)*, is an automorphism of* M*. This uses Remark 2.1*

Let N be the structure $(\mathbb{R}, <, \sigma', P')$, where $<$ is the usual order on $\mathbb{R}$, $\sigma'(x) = x + \pi$, and $P'(x, y_1, \ldots, y_4)$ holds if and only if $x < y_i < x + \pi$ for each i and

$$\cot(y_1 - x) - \cot(y_2 - x) = \cot(y_3 - x) - \cot(y_4 - x).$$

Remark 2.4. *The structure* N *is isomorphic to the structure* M *via the map sending* x *to* $(\lfloor x/\pi \rfloor, -\cot(x))$*, using Remark 2.2.*

Remark 2.5. *For every* $\alpha \in \mathbb{R}$*, the map* $x \mapsto x + \alpha$ *is an automorphism of* N*. Consequently, the automorphism group of* N *acts transitively on* N *and the same is true for* M*.*

One thinks of the structure M as being the "universal cover" of $(\mathbb{RP}^1, P_0)$. We will show that M is o-minimal and fails condition (d) of Lemma 1.2.

2.1 O-minimality

Consider the two-sorted structure $(\mathbb{R}, \mathbb{Z}, \ldots)$ with the ring structure on $\mathbb{R}$ and the order on $\mathbb{Z}$. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ is *not* definable in this structure; the two sorts $\mathbb{R}$ and $\mathbb{Z}$ have nothing to do with each other.

Remark 2.6. *The structure* M *can be interpreted in* $(\mathbb{R}, \mathbb{Z}, \ldots)$*, by mapping* $(n, x) \in \mathbb{Z} \times \mathbb{R}$ *to* (n, x) *and* (n, ∞) *to* $n \in \mathbb{Z}$*.*

We draw two consequences from this:

Lemma 2.7. *Let* D *be a definable subset of* M^1*, and suppose* $a, b \in M$*. Then* $D \cap [a, b]$ *is a finite union of points and intervals.*

Proof. In the structure $(\mathbb{R}, \mathbb{Z}, \ldots)$, the set $\mathbb{R}$ is o-minimal. Under the interpretation of M in $(\mathbb{R}, \mathbb{Z}, \ldots)$, each open interval of the form $\{n\} \times \mathbb{R} \subset \mathbb{Z} \times \mathbb{RP}^1 = M$ is in definable bijection with $\mathbb{R}$. Consequently, $D \cap (\{n\} \times \mathbb{R})$ is a finite union of points and intervals. More generally, each interval $[a, b] \subset M$ is contained in a finite union of points and open intervals of the form $\{n\} \times \mathbb{R}$, so the conclusion holds. $\square$

Corollary 2.8. *Let $M + M$ be the structure obtained by laying two copies of M end-to-end. More precisely, $M + M$ is the structure $(2 \times M, <, \sigma, P)$, where*

- *$(2 \times M, <)$ is $\{1, 2\} \times M$ with the lexicographic ordering.*
- *$\sigma(i, x) = (i, \sigma(x))$ for $i = 1, 2$.*
- *$P((i_1, x_1), \ldots, (i_5, x_5))$ agrees with $P(x_1, \ldots, x_5)$ when $i_1 = i_2 = \cdots = i_5$, and is false otherwise.*

Then the two inclusion maps $\iota_1, \iota_2 : M \to M + M$ are elementary embeddings.

Proof. The two canonical inclusion maps of the ordered set $\mathbb{Z}$ into the ordered set $\mathbb{Z} + \mathbb{Z} := 2 \times \mathbb{Z}$ are both elementary embeddings. This is an easy exercise using quantifier elimination in $(\mathbb{Z}, <, \sigma)$, where $\sigma(n) = n + 1$. From this, it follows that the two canonical inclusions

$$(\mathbb{R}, \mathbb{Z}, \ldots) \hookrightarrow (\mathbb{R}, \mathbb{Z} + \mathbb{Z}, \ldots)$$

are elementary embeddings. Applying the interpretation of M in $(\mathbb{R}, \mathbb{Z}, \ldots)$ to both sides yields the desired result. $\square$

Remark 2.9. *If τ_1, τ_2 are two automorpisms of M, then the map on $M + M$ which acts as τ_1 on the first copy and τ_2 on the second copy is an automorphism.*

Remark 2.10. *The group of automorphisms of $M+M$ which fix the first copy pointwise acts transitively on the second copy. This follows from Remarks 2.5 and 2.9. As a consequence, if D is a one-dimensional definable subset, defined over the first copy, then D or its complement contains the second copy.*

Theorem 2.11. *The structure M is o-minimal.*

Proof. The structure N is interpretable in $\mathbb{R}$ with the ring structure and with the trigonometric functions restricted to the interval $[0, \pi]$. This is known to be o-minimal.

Alternatively, here is a more elementary argument:

- $M \cong N$ has the order type of $\mathbb{R}$, so it suffices to show that if $D \subset M$ is definable, then the boundary ∂D does not accumulate at any points in the extended line $\{-\infty\} \cup M \cup \{+\infty\}$.
- Lemma 2.7 shows that ∂D cannot accumulate at any points in M.
- Suppose ∂D had an accumulation point at $+\infty$. Then ∂D is not bounded above. This remains true in the elementary extension $M + M$, where we identify the original M with the first copy in $M + M$. But by Remark 2.10, D or its complement contains the second copy, making ∂D disjoint from the second copy. Then ∂D is bounded above by any element from the second copy, a contradiction.
- A similar argument shows that ∂D has no accumulation point at $-\infty$.

$\square$

2.2 Failure of elimination of imaginaries

Consider the structure $M + M$ from Corollary 2.8. Call the two copies M_1 and M_2. Each is isomorphic to M, and each is an elementary substructure of $M_1 + M_2$. We will show that condition (d) of Lemma 1.2 fails in M_2. Suppose for the sake of contradiction that the elementary diagram of M_2 eliminates imaginaries.

Let $\alpha = (0, \infty) \in M_1$. Let X be the α-definable set

$$X = \{(x, y) : \alpha < x < y < \sigma(\alpha)\}$$

We can identify the open interval $(\alpha, \sigma(\alpha)) = \{0\} \times \mathbb{R}$ with $\mathbb{R}$. Then X is identified with $\{(x, y) \in \mathbb{R}^2 : x < y\}$. Let $\sim$ be the relation on X

$$(x, y) \sim (x', y') \iff P(\alpha, x, y, x', y').$$

Under the identification of the open interval $(\alpha, \sigma(\alpha))$ with $\mathbb{R}$, we have

$$(x, y) \sim (x', y') \iff P_0(\infty, x, y, x', y') \iff x - y = x' - y'. \tag{1}$$

Thus $\sim$ is an equivalence relation on X.

By Remarks 2.3 and 2.9, for each $a > 0$ and $b \in \mathbb{R}$, there is an automorphism $\tau_{a,b}$ of the structure $M_1 + M_2$ which sends (n, x) to $(n, ax + b)$ on M_1, and which fixes M_2 pointwise. Note that $\tau_{a,b}$ fixes α, and therefore acts on the α-definable quotient $X/\sim$. Identifying X with $\{(x, y) \in \mathbb{R}^2 : x < y\}$, we see that

$$\tau_{a,b}(x, y) \sim (x, y) \iff \tau_{a,b}(x) - \tau_{a,b}(y) = x - y \iff ax - ay = x - y \iff a = 1.$$

So if $a = 1$, then $\tau_{a,b}$ acts trivially on $X/\sim$, and otherwise, $\tau_{a,b}$ has no fixed points.

Let c be any element of $X/\sim$. Under the assumption that the elementary diagram of M_2 eliminates imaginaries, c is interdefinable over $M_2\alpha$ with some subset $S \subset M_1$. Note that $\tau_{a,b}$ fixes $M_2\alpha$ pointwise, so $\tau_{a,b}$ fixes c if and only if it fixes S pointwise. In particular $\tau_{1,1}$ fixes c and $\tau_{2,0}$ does not, so S must be fixed pointwise by $\tau_{1,1}$, but not by $\tau_{2,0}$. This is impossible, however, since the action of $\tau_{1,1}$ on M_1 is $(n, x) \mapsto (n, x + 1)$. The only fixed points are of the form (n, ∞), and these are also fixed by $\tau_{2,0}$, the map sending $(n, x) \mapsto (n, 2x)$. So if S is fixed pointwise by $\tau_{1,1}$, it is also fixed pointwise by $\tau_{2,0}$, a contradiction.

So the equivalent conditions of Lemma 1.2 fail in the o-minimal structure M.

Remark 2.12. *The quotient $X/\sim$ described above can be eliminated by naming parameters from M_1. This quotient is a counterexample to (d) of Lemma 1.2, rather than to (a). Tracing through Lemma 1.2, the actual quotient in M which cannot be eliminated is $Y/\approx$, where $Y \subset M^3$ is the set of (a, b, c) such that $a < b < c < \sigma(a)$, and where*

$$(a, b, c) \approx (a', b', c') \iff a = a' \wedge P(a, b, c, b', c').$$

3 Local definability

Unlike Question 1.1, Question 1.3 has an easy affirmative answer.

Lemma 3.1. *Given an o-minimal structure M and an imaginary $e \in M^{eq}$, there is a set $A \subset M$ and a real tuple $c \in M^k$ such that $A \mathop{\smile\hskip-0.9em\raise0.9ex\hbox{$\vert$}}^{\text{þ}} e$ and $\operatorname{dcl}^{eq}(Ae) = \operatorname{dcl}^{eq}(Ac)$.*

Proof. Suppose e is a class of the definable equivalence relation E. Let x be some representative of this class. So x is a (real) tuple, and $e \in \operatorname{dcl}^{eq}(x)$. Consider the pregeometry on M coming from definability over e, i.e., the pregeometry where the closure of a set $S \subset M$ is $M \cap \operatorname{dcl}^{eq}(Se)$. Let $A \subset x$ be a basis for x, and let c be the remaining coordinates of x. Then $x = Ac$, and so

$$e \in \operatorname{dcl}^{eq}(x) = \operatorname{dcl}^{eq}(Ac).$$

Also, since A is a basis for x, $c \subset x$ is in the closure of A:

$$c \in \operatorname{dcl}^{eq}(Ae).$$

Finally, note that $rank(x/e) = rank(A/e) = |A|$ because A is a basis over e. Since A has size $|A|$ and is made of singletons,

$$rank(A/\emptyset) \leq |A|.$$

On the other hand

$$rank(A/\emptyset) \geq rank(A/e) = |A|$$

on general grounds. So $rank(A/\emptyset) = rank(A/e)$, which implies $A \mathop{\smile\hskip-0.9em\raise0.9ex\hbox{$\vert$}}^{\text{þ}} e$. $\square$

The affirmative answer to Question 1.3 does not imply an affirmative answer to Question 1.1. The implication fails because the auxiliary parameters A in Lemma 3.1 depend too strongly on e. Lemma 3.1 can be vaguely interpreted as saying that interpretable sets look "locally" like definable sets.

This idea is made more precise by the following results, whose proofs we omit.

Theorem 3.2. *Suppose $(M, <)$ is an o-minimal expansion of a dense linear order. Let $Y \subset M^n$ be a definable set, and E be a definable equivalence relation on Y. Then there is a relatively open definable set $Y' \subset Y$, with*

$$\dim(Y \setminus Y') < \dim Y,$$

such that the quotient topology on Y'/E is definable, Hausdorff, and locally Euclidean. That is, Y'/E is Hausdorff and has a definable basis of opens which are homeomorphic to open subsets of M^n for various n. Moreover, if Y' is chosen sufficiently small, then for any smaller open $Y'' \subset Y'$ with $\dim(Y'' \setminus Y) < \dim Y$, the quotient topology on Y''/E will have these same properties, and $Y''/E \hookrightarrow Y'/E$ will be an open immersion.

The "moreover" clause gives some degree of uniqueness to the topology on the quotient.

Corollary 3.3. *If X is an interpretable set in a (dense) o-minimal theory, then there is a definable topology on X which makes X be locally Euclidean and Hausdorff, and have finitely many definably connected components.*

Without the assumption of finitely many definably connected components, this corollary be trivial: we could take the discrete topology on X.

The strategy for proving Theorem 3.2 is to arrange that the map from Y' to the quotient topological space Y'/E is an open map (images of open sets are open). This condition is definable, and it ensures that the quotient topology on Y'/E is definable. It also ensures that $Y''/E \hookrightarrow Y'/E$ is an open immersion for open subsets $Y'' \subset Y'$, which allows us to shrink Y to Y' in several steps, maintaining previously obtained properties of the quotient at each stage.

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